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Uniform decay estimates for finite-energy solutions of semi-linear elliptic inequalities and geometric applications

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ABSTRACT

We prove uniform decay estimates at infinity for solutions $0 \leqslant u \in L^p$ of the semilinear elliptic inequality $\Delta u + au^\sigma + bu \geqslant 0$, $a, b \geqslant 0$, $\sigma \geqslant 1$, in the presence of a Sobolev inequality (with potential term). This gives a unified point of view in the investigation of different geometric questions. In particular, we present applications to the study of the topology at infinity of parallel mean curvature submanifolds, to the non-compact Yamabe problem, and to estimate the decay rate of the traceless Ricci tensor of conformally flat manifolds.

1. Introduction

The main achievement of this note is represented by the following general estimating result. Roughly speaking, in the spirit of Moser iteration methods, it shows how to obtain the rate of uniform decay at infinity for solutions $u \geqslant 0$ of the semilinear elliptic inequality $\Delta u + au^\sigma + bu \geqslant 0$, $a, b \geqslant 0$, $\sigma \geqslant 1$, from suitable integral estimates of u, provided: (a) the underlying manifold M supports a Sobolev inequality (possibly with potential term) and (b) the solution u (which in general has a geometric content) suitably rescales whenever the background metric rescales by a constant factor.

Theorem 1. *Let $(M, \langle\,,\rangle)$ be an m-dimensional, complete Riemannian manifold satisfying, outside a compact set $K \subset M$, the Sobolev inequality*

$$\left(\int_M \varphi^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leqslant A^2 \int_M |\nabla\varphi|^2 + B^2 \int_M \varphi^2 \tag{1}$$

for every $\varphi \in C_c^\infty(M \setminus K)$, and for some constants $A > 0$, $B \geqslant 0$. Let $0 \leqslant u \in Lip_{\mathrm{loc}}(M)$ be a function depending on the background metric of M in such a way that, under the dilation

$$\widetilde{\langle\,,\rangle} = \lambda^2 \langle\,,\rangle, \quad \lambda = const. > 0, \tag{2}$$

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it holds

$$\tilde{u} = \frac{u}{\lambda^q}, \tag{3}$$

for some $q \geqslant 1$. Assume that u satisfies the differential inequality

$$\Delta u \geqslant -au^\sigma - bu, \tag{4}$$

weakly on $M \setminus K$, where a, b, σ are real numbers such that

$$a > 0, \qquad b \geqslant 0, \qquad 1 \leqslant \sigma \leqslant \frac{q+2}{q}. \tag{5}$$

Set

$$\xi = \begin{cases} 0, & b+B>0, \\ \frac{(\sigma-1)m}{2}, & b+B=0. \end{cases}$$

If

$$\limsup_{R\to+\infty} \left\{ R^{p(2\gamma-\xi)} \int\limits_{M\setminus B_R} u^{2\gamma} \right\} = 0 \quad (\textit{resp.} \ < +\infty), \tag{6}$$

for some $0 \leqslant p \leqslant q$ and $\gamma \geqslant \frac{m}{2q}$, then

$$\limsup_{R\to\infty} \left\{ R^p \sup_{M\setminus B_R} u \right\} = 0 \quad (\textit{resp.} \ < +\infty, \ \textit{provided} \ 2\gamma \neq \xi). \tag{7}$$

The rescaling assumption on the function u could sound quite strange but, in fact, it is natural in geometric applications. Indeed u encodes geometric data, such as the length of the mean curvature of an isometric immersion or the stretching factor in the conformal deformation, and therefore it depends directly on the background metric; see also Remark 3 below.

We also explicitly note that, under the assumption

$$u \in L^{2\gamma}(M), \tag{8}$$

for some $\gamma \geqslant \frac{m}{2q}$, an application of Theorem 1 with $p = 0$ gives the uniform estimate

$$\lim_{R\to+\infty} \sup_{M\setminus B_R} u = 0.$$

As a matter of fact, in the special case

$$b = 0, \qquad \gamma = \frac{m}{2q}, \qquad B = 0, \qquad \sigma = \frac{q+2}{q},$$

(8) implies the improved conclusion

$$\lim_{R\to+\infty} R^q \sup_{M\setminus B_R} u = 0.$$

Indeed, in this situation, the integral estimate (6) is trivially met. For instance, this occurs if the Sobolev inequality (1) holds with $B = 0$, $\sigma = (m+2)/(m-2)$ and u rescales by a factor $q = (m-2)/2$. As we shall see in Section 4.2 below, this happens for solutions of the Yamabe problem. A similar situation occurs for minimal submanifolds in the Euclidean space; see Remark 15 in Section 4.1.

As expected, conditions (6) and (7) show that the rate of uniform decay at infinity of the solution u (e.g. in case $b \neq 0$) is influenced by the integral decay rate of u. Therefore, a natural and very interesting problem is to determine which geometric conditions, if any, imply the validity of (5). We shall consider this problem in Section 3 below.

Theorem 1 stems from a nice paper by Y.B. Shen and X.H. Zhu [18]; see also the extensions by S. Xu and Q. Deng [19]. These papers are essentially devoted to the study of uniform curvature estimates for parallel mean curvature submanifolds, with finite total curvature, into space-forms. Using a quite different approach, similar estimates (and some companion geometric consequences) were previously obtained by P. Berard, M. do Carmo and W. Santos [3], and M. do Carmo, L.F. Cheung and W. Santos [7].

According to its own origin, Theorem 1 can be specified to the parallel-mean-curvature setting to get uniform curvature estimates which sharpen and extend those in [18,19,3,7]. These estimates turn out to be very useful in the study of the topology at infinity of parallel mean curvature submanifolds, with finite total curvature, into space-forms of non-positive sectional curvature. In particular they can be used to deduce information on the number of ends of the submanifold and on the volume growth of each end. We will deal with these topics in Section 4.1 below.

The range of application of Theorem 1 goes beyond the submanifolds theory and, in fact, Theorem 1 can be used to study in a unified way geometric problems that can be summarized into a finite energy solution of a semilinear elliptic inequality of the type (4). For instance, inequality (4) naturally guides us towards the Yamabe realm and Theorem 1 enables us to prove estimating results for, e.g., the energy density of conformal immersions of a locally conformally flat manifold with bounded scalar curvature into equidimensional spheres, as explained in Section 4.2. In this setting it should be noted that the integrability assumption may give geometric constraints on the conformally deformed metric; see Proposition 24.

In the setting of locally conformally flat manifolds of asymptotically constant scalar curvature, another interesting application of Theorem 1 concerns the decay rate at infinity of the traceless Ricci tensor. This example will be considered in the last section of the paper; see Section 4.3.

Before entering the main body of the paper, we observe that it would be very interesting to extend Theorem 1 and its geometric consequences to the case where the coefficients a and b are possibly unbounded functions (with controlled growth). We shall investigate these possible extensions in a subsequent paper.

2. Proof of the uniform decay estimates

In this section we give a proof of the main analytic result of the paper. In fact, we shall obtain Theorem 1 as a consequence of the following more general local estimates.

Theorem 2. *Keeping the notations of Theorem 1, let $R_2 > 3R_1 > 0$ and $\delta > 0$ be such that B_{R_1} contains the compact set K and*

$$\sup_{B_{R_2-R_1}\setminus B_{2R_1}} u \geqslant \delta > 0. \tag{9}$$

Fix

$$\gamma \geqslant \frac{m}{2q}.$$

If, for some $\varepsilon_0 > 0$ and $r > 0$,

$$\int_{B_{R_2}\setminus B_{R_1}} u^{2\gamma} \leqslant \varepsilon_0 \tag{10}$$

and

$$r^q \sup_{B_{R_2-r}\setminus B_{R_1+r}} u \geqslant \varepsilon_0^{-\frac{1}{2\gamma}} \left(\int_{B_{R_2}\setminus B_{R_1}} u^{2\gamma} \right)^{\frac{1}{2\gamma}}, \tag{11}$$

then there exists a constant $C > 0$, depending only on m, A, B, a, b, δ and γ, such that

$$\varepsilon_0 \geqslant C.$$

Notation. We begin by fixing some notation that will be used throughout the next proofs. For any positive rays $R_2 > R_1 > 0$ and $0 \leqslant r < \frac{R_2-R_1}{2}$, we define $A(R_1,R_2) := B_{R_2} \setminus \overline{B_{R_1}}$ and $BT(r,R_1,R_2) := \overline{A(R_1+r,R_2-r)}$. We also set $\|v\|_{p,\Omega} = (\int_\Omega |v|^p)^{1/p}$.

Proof of Theorem 2. Let us choose $r_0 \in [0,R_1]$ and $Q_0 \in BT(r_0,R_1,R_2)$ such that

$$r_0^q \sup_{BT(r_0,R_1,R_2)} u = \max_{0\leqslant r\leqslant R_1} \left\{ r^q \sup_{BT(r,R_1,R_2)} u \right\} \tag{12}$$

and

$$u_0 := u(Q_0) = \max_{BT(r_0,R_1,R_2)} u. \tag{13}$$

From (9) and (13) we get

$$u_0 = \sup_{BT(r_0,R_1,R_2)} u \geqslant \sup_{A(2R_1,R_2-R_1)} u \geqslant \delta > 0.$$

Moreover, by choosing α so that $0<\alpha<1-2^{-\frac{1}{q}}$, by (12) we have

$$\sup_{BT(r_0,R_1,R_2)} u=\frac{1}{r_0^q}\max_{0\leqslant r\leqslant R_1}\Big\{r^q\sup_{BT(r,R_1,R_2)} u\Big\}\geqslant \alpha^q \sup_{BT(\alpha r_0,R_1,R_2)} u.$$

By noticing that $B_{2^{-\frac{1}{q}}r_0}(Q_o)\subseteq BT(\alpha r_0,R_1,R_2)$, this latter yields

$$\sup_{B_{2^{-\frac{1}{q}}r_0}(Q_0)} u\leqslant \sup_{BT(\alpha r_0,R_1,R_2)} u\leqslant \alpha^{-q}\sup_{BT(r_0,R_1,R_2)} u=\alpha^{-q}u_0. \tag{14}$$

Set

$$\zeta:=\int_{A(R_1,R_2)} u^{2\gamma}\,dv_M$$

and note that, by (10), $\zeta\leqslant\varepsilon_0$. Then (11) implies

$$r_0^q u_0\geqslant \varepsilon_0^{-\frac{1}{2\gamma}}\zeta^{\frac{1}{2\gamma}}=\big(\varepsilon_0^{-1}\zeta\big)^{\frac{1}{2\gamma}}. \tag{15}$$

We consider a new metric obtained by rescaling the original metric $\langle,\rangle$ as follows

$$\widetilde{\langle,\rangle}=\big[2\big(\varepsilon_0\zeta^{-1}\big)^{\frac{1}{2\gamma}}u_0\big]^{\frac{2}{q}}\langle,\rangle.$$

From (15) we get

$$\begin{aligned}\tilde B_1(Q_0)&=\big\{x\in M:\ \tilde d(x,Q_0)<1\big\}\\&=\big\{x\in M:\ d(x,Q_0)<(2u_o)^{-\frac{1}{q}}\big(\varepsilon_0^{-1}\zeta\big)^{\frac{1}{2\gamma q}}<2^{-\frac{1}{q}}r_0\big\}\\&\subseteq B_{2^{-\frac{1}{q}}r_0}(Q_0).\end{aligned} \tag{16}$$

By the assumptions on u, from (14) and (16) and noting that

$$\tilde u=\big[(2u_0)^{-1}\big(\varepsilon_0^{-1}\zeta\big)^{\frac{1}{2\gamma}}\big]u$$

we obtain

$$\begin{aligned}\sup_{\tilde B_1(Q_0)}\tilde u&\leqslant\big[(2u_0)^{-1}\big(\varepsilon_0^{-1}\zeta\big)^{\frac{1}{2\gamma}}\big]\sup_{B_{2^{-\frac{1}{q}}r_0}(Q_0)} u\\&\leqslant\frac{\alpha^{-q}}{2}\big(\varepsilon_0^{-1}\zeta\big)^{\frac{1}{2\gamma}}=\alpha^{-q}\big[(2u_0)^{-1}\big(\varepsilon_0^{-1}\zeta\big)^{\frac{1}{2\gamma}}\big]u_o=\alpha^{-q}\tilde u_0.\end{aligned}$$

Thus

$$\sup_{\tilde B_1(Q_0)}\tilde u\leqslant\frac{\alpha^{-q}}{2}=:C_1; \tag{17}$$

$$\big(\varepsilon_0^{-1}\zeta\big)^{\frac{1}{2\gamma}}=2\tilde u_0. \tag{18}$$

Moreover, applying (9) and (16), the rescaling property of u and assumption $q\geqslant\frac{m}{2\gamma}$ give

$$\begin{aligned}\int_{\tilde B_1(Q_0)}\tilde u^{2\gamma}\,d\tilde v_M&\leqslant\int_{B_{2^{-\frac{1}{q}}r_0}(Q_0)} u^{2\gamma}\,dv_M\big[(2u_0)^{-1}\big(\varepsilon_0^{-1}\zeta\big)^{\frac{1}{2\gamma}}\big]^{(2\gamma-\frac{m}{q})}\\&\leqslant C_2\int_{BT(\alpha r_0,R_1,R_2)} u^{2\gamma}\,dv_M\leqslant C_2\zeta\end{aligned} \tag{19}$$

where $C_2:=(2\delta)^{(\frac{m}{q}-2\gamma)}>0$. Observe that, using (1) and (9) and computing the rescaled quantities $d\tilde v_M$ and $|\tilde\nabla\cdot|$, we have

$$\begin{aligned}\left(\int_M \varphi^{\frac{2m}{m-2}}\, d\tilde{v}_M\right)^{\frac{m-2}{m}} &\leqslant A^2 \int_M |\tilde{\nabla}\varphi|^2\, d\tilde{v}_M + B^2\big[2\big(\varepsilon_0\zeta^{-1}\big)^{\frac{1}{2\gamma}} u_0\big]^{-\frac{2}{q}} \int_M \varphi^2\, d\tilde{v}_M \\ &\leqslant A^2 \int_M |\tilde{\nabla}\varphi|^2\, d\tilde{v}_M + B^2(2\delta)^{-\frac{2}{q}} \int_M \varphi^2\, d\tilde{v}_M \end{aligned} \tag{20}$$

for every $\varphi \in C_c^\infty(M \setminus K)$.

Finally, because of (3), (9), (17) and the laplacian transformation rule under rescaling of the metric, we get

$$\begin{aligned}\tilde{\Delta}\tilde{u} &= \big[(2u_0)^{-1}\big(\varepsilon_0^{-1}\zeta\big)^{\frac{1}{2\gamma}}\big]^{\frac{q+2}{q}} \Delta u \\ &\geqslant \big[(2u_0)^{-1}\big(\varepsilon_0^{-1}\zeta\big)^{\frac{1}{2\gamma}}\big]^{\frac{q+2}{q}} \big(-au^\sigma - bu\big) \\ &= -a\big[(2u_0)^{-1}\big(\varepsilon_0^{-1}\zeta\big)^{\frac{1}{2\gamma}}\big]^{\frac{q+2-\sigma q}{q}} \tilde{u}^\sigma - b\big[(2u_0)^{-1}\big(\varepsilon_0^{-1}\zeta\big)^{\frac{1}{2\gamma}}\big]^{\frac{2}{q}} \tilde{u} \geqslant -\rho\tilde{u} \end{aligned} \tag{21}$$

on $\tilde{B}_1(Q_0)$, with

$$\rho := \big[a(2\delta)^{\frac{\sigma q-q-2}{q}} C_1^\sigma + b(2\delta)^{-\frac{2}{q}}\big] \sim \begin{cases} \delta^{-2/q}, & b>0, \\ \delta^{\sigma-1-2/q}, & b=0, \end{cases} \qquad \delta \ll 1.$$

Now, let $\eta \in C_c^1(\tilde{B}_1(Q_0))$ be chosen later and let X be the vector field defined, for $k>1$, by

$$X := \eta^2 \tilde{u}^{2k-1} \tilde{\nabla}\tilde{u}.$$

Then, applying the Stokes theorem, we deduce (in what follows, to simplify the writing, we omit the symbol $\tilde{}$ over norms and scalar products)

$$\begin{aligned} 0 &= \int_{\tilde{B}_1(Q_0)} \operatorname{div} X\, d\tilde{v}_M \\ &= 2\int_{\tilde{B}_1(Q_0)} \eta\tilde{u}^{2k-1}\langle\tilde{\nabla}\tilde{u}, \tilde{\nabla}\eta\rangle\, d\tilde{v}_M + \int_{\tilde{B}_1(Q_0)} \eta^2\big\langle\tilde{\nabla}\tilde{u}^{2k-1}, \tilde{\nabla}\tilde{u}\big\rangle d\tilde{v}_M + \int_{\tilde{B}_1(Q_0)} \eta^2\tilde{u}^{2k-1}\tilde{\Delta}\tilde{u}\, d\tilde{v}_M \\ &\geqslant 2\int_{\tilde{B}_1(Q_0)} \left\langle \frac{1}{\sqrt{2}}\eta\tilde{u}^{k-1}\tilde{\nabla}\tilde{u}, \sqrt{2}\tilde{u}^k\tilde{\nabla}\eta\right\rangle d\tilde{v}_M + \int_{\tilde{B}_1(Q_0)} \eta^2(2k-1)\tilde{u}^{2k-2}|\tilde{\nabla}\tilde{u}|^2\, d\tilde{v}_M - \rho\int_{\tilde{B}_1(Q_0)} \eta^2\tilde{u}^{2k}\, d\tilde{v}_M \\ &\geqslant -\frac{1}{2}\int_{\tilde{B}_1(Q_0)} \eta^2\tilde{u}^{2k-2}|\tilde{\nabla}\tilde{u}|^2\, d\tilde{v}_M - 2\int_{\tilde{B}_1(Q_0)} \tilde{u}^{2k}|\tilde{\nabla}\eta|^2\, d\tilde{v}_M + (2k-1)\int_{\tilde{B}_1(Q_0)} \eta^2\tilde{u}^{2k-2}|\tilde{\nabla}\tilde{u}|^2\, d\tilde{v}_M - \rho\int_{\tilde{B}_1(Q_0)} \eta^2\tilde{u}^{2k}\, d\tilde{v}_M \end{aligned}$$

which, in turn, implies

$$\left(2k-\frac{3}{2}\right)\int_{\tilde{B}_1(Q_0)} \eta^2\tilde{u}^{2k-2}|\tilde{\nabla}\tilde{u}|^2\, d\tilde{v}_M \leqslant 2\int_{\tilde{B}_1(Q_0)} \tilde{u}^{2k}|\tilde{\nabla}\eta|^2\, d\tilde{v}_M + \rho\int_{\tilde{B}_1(Q_0)} \eta^2\tilde{u}^{2k}\, d\tilde{v}_M. \tag{22}$$

On the other hand, using also the elementary inequality $2ab \leqslant (\sqrt{2}a)^2 + (\frac{b}{\sqrt{2}})^2$ for all $a, b \in \mathbb{R}$, we have

$$\begin{aligned}\int_{\tilde{B}_1(Q_0)} \big|\tilde{\nabla}\big(\eta\tilde{u}^k\big)\big|^2\, d\tilde{v}_M &= k^2\int_{\tilde{B}_1(Q_0)} \eta^2\tilde{u}^{2k-2}|\tilde{\nabla}\tilde{u}|^2\, d\tilde{v}_M + \int_{\tilde{B}_1(Q_0)} \tilde{u}^{2k}|\tilde{\nabla}\eta|^2\, d\tilde{v}_M + 2k\int_{\tilde{B}_1(Q_0)} \eta\tilde{u}^{2k-1}\langle\tilde{\nabla}\tilde{u}, \tilde{\nabla}\eta\rangle\, d\tilde{v}_M \\ &\leqslant k\left(k+\frac{1}{2}\right)\int_{\tilde{B}_1(Q_0)} \eta^2\tilde{u}^{2k-2}|\tilde{\nabla}\tilde{u}|^2\, d\tilde{v}_M + (1+2k)\int_{\tilde{B}_1(Q_0)} \tilde{u}^{2k}|\tilde{\nabla}\eta|^2\, d\tilde{v}_M. \end{aligned}$$

Using (22) into the above we obtain

$$\begin{aligned}\int_{\tilde{B}_1(Q_0)} \big|\tilde{\nabla}\big(\eta\tilde{u}^k\big)\big|^2\, d\tilde{v}_M &\leqslant \frac{k(2k+1)}{(4k-3)}\rho\int_{\tilde{B}_1(Q_0)} \eta^2\tilde{u}^{2k}\, d\tilde{v}_M + \frac{3(4k^2-1)}{4k-3}\int_{\tilde{B}_1(Q_0)} \tilde{u}^{2k}|\tilde{\nabla}\eta|^2\, d\tilde{v}_M \\ &\leqslant kC_3\int_{\tilde{B}_1(Q_0)} \big(|\tilde{\nabla}\eta|^2 + \eta^2\big)\tilde{u}^{2k}\, d\tilde{v}_M \end{aligned}$$

with

$$\begin{aligned} C_3 &= \max\left\{\frac{3(4k^2-1)}{k(4k-3)}; \frac{2k+1}{(4k-3)}\rho\right\} \\ &\leqslant \max\{9; 3\rho\} \sim \begin{cases} \delta^{-2/q}, & b>0, \\ \delta^{\sigma-1-2/q}, & b=0, \end{cases} \quad \delta \ll 1. \end{aligned}$$

Applying the rescaled Sobolev inequality (20) to the function $\eta\tilde{u}^k$ we get

$$\begin{aligned} \|\eta\tilde{u}^k\|^2_{\frac{2m}{m-2}} &= \Bigg[\int\limits_{\tilde{B}_1(Q_0)} \left(\eta^2\tilde{u}^{2k}\right)^{\frac{m}{m-2}} d\tilde{v}_M\Bigg]^{\frac{m-2}{m}} \\ &\leqslant A^2 \int\limits_{\tilde{B}_1(Q_0)} \left|\tilde{\nabla}\left(\eta\tilde{u}^k\right)\right|^2 d\tilde{v}_M + (2\delta)^{-\frac{2}{q}} B^2 \int\limits_{\tilde{B}_1(Q_0)} \eta^2\tilde{u}^{2k}\, d\tilde{v}_M \\ &\leqslant A^2 k C_3 \int\limits_{\tilde{B}_1(Q_0)} \left(|\tilde{\nabla}\eta|^2+\eta^2\right)\tilde{u}^{2k}\, d\tilde{v}_M + (2\delta)^{-\frac{2}{q}} B^2 \int\limits_{\tilde{B}_1(Q_0)} \eta^2\tilde{u}^{2k}\, d\tilde{v}_M \\ &\leqslant kC_4\left\{\left\||\tilde{\nabla}\eta|\tilde{u}^k\right\|_2^2 + \left\|\eta\tilde{u}^k\right\|_2^2\right\} \end{aligned}$$

for all $\eta \in C^1_c(\tilde{B}_1(Q_0))$ and $\forall k>1$, with

$$\begin{aligned} C_4 &= A^2C_3 + \frac{(2\delta)^{-\frac{2}{q}}B^2}{k} \\ &< A^2C_3 + (2\delta)^{-\frac{2}{q}}B^2 \sim \begin{cases} \delta^{-\frac{2}{q}}, & b+B>0, \\ \delta^{-\frac{2}{q}(1-\frac{(\sigma-1)q}{2})}, & b+B=0, \end{cases} \quad \delta \ll 1. \end{aligned}$$

We are now in the position to perform the standard Nash–Moser iteration procedure.

Let $R_i := \frac{3}{2} - \sum_{j=0}^{i} 2^{-j-1}$, so that $R_0 = 1$ and $R_i \to \frac{1}{2}$ for $i \to \infty$. Define $B_i := \tilde{B}_{R_i}(Q_0)$. Choose functions $\eta_i \in C^1_c(\tilde{B}_1(Q_0))$ such that $0 \leqslant \eta_i \leqslant 1$, $|\tilde{\nabla}\eta_i| \leqslant 2^{i+3}$, $\operatorname{supp}(\eta_i) \subset B_i$ and $\eta_i|_{B_{i+1}} \equiv 1$; finally set $\chi_i := \chi_{\operatorname{supp}\eta_i} \geqslant \eta_i$, the characteristic function of the set $\operatorname{supp}\eta_i$. We have

$$\|\eta_i\tilde{u}^k\|^2_{\frac{2m}{m-2}} \leqslant C_4k\left\{\left\||\tilde{\nabla}\eta_i|\tilde{u}^k\right\|_2^2 + \left\|\eta_i\tilde{u}^k\right\|_2^2\right\} \leqslant C_4k\left\{\left(1+2^{2i+4}\right)\left\|\chi_i\tilde{u}^k\right\|_2^2\right\},$$

from which

$$\Bigg(\int\limits_{B_{i+1}} \tilde{u}^{\frac{2mk}{m-2}}\, d\tilde{v}_M\Bigg)^{\frac{m-2}{m}} \leqslant 17C_4 4^i k\Bigg(\int\limits_{B_i} \tilde{u}^{2k}\, d\tilde{v}_M\Bigg).$$

Set $k_i := \gamma(\frac{m}{m-2})^i$. Thus

$$\begin{aligned} \|\tilde{u}\|_{2k_{i+1},B_{i+1}} &= \Bigg(\int\limits_{B_{i+1}} \tilde{u}^{2k_{i+1}}\, d\tilde{v}_M\Bigg)^{\frac{1}{2k_{i+1}}} = \Bigg(\int\limits_{B_{i+1}} \tilde{u}^{\frac{2mk_i}{m-2}}\, d\tilde{v}_M\Bigg)^{\frac{m-2}{2mk_i}} \\ &\leqslant \Bigg[17C_4 4^i k_i\Bigg(\int\limits_{B_i} \tilde{u}^{2k_i}\, d\tilde{v}_M\Bigg)\Bigg]^{\frac{1}{2k_i}} \\ &= \Bigg[17C_4\gamma\left(\frac{4m}{m-2}\right)^i\Bigg]^{\frac{1}{2\gamma}(\frac{m-2}{m})^i} \|\tilde{u}\|_{2k_i,B_i}. \end{aligned} \tag{23}$$

Set

$$C_5 := [17C_4\gamma]^{\frac{1}{2\gamma}} \sim \begin{cases} \delta^{-\frac{1}{\gamma q}}, & b+B>0, \\ \delta^{-\frac{(1-(\sigma-1)q/2)}{\gamma q}}, & b+B=0, \end{cases} \quad \delta \ll 1,$$

and $C_6 := [\frac{4m}{m-2}]^{\frac{1}{2\gamma}}$, so that

$$\left[17C_4\gamma\left(\frac{4m}{m-2}\right)^i\right]^{\frac{1}{2\gamma}(\frac{m-2}{m})^i} = C_5^{(\frac{m-2}{m})^i} C_6^{i(\frac{m-2}{m})^i} = e^{(\ln C_5)(\frac{m-2}{m})^i + (\ln C_6) i (\frac{m-2}{m})^i}.$$

Then iterating (23), we get

$$\begin{aligned}\|\tilde{u}\|_{\infty, \tilde{B}_{\frac{1}{2}}(Q_0)} &= \lim_{i\to+\infty} \|\tilde{u}\|_{2k_{i+1}, B_{i+1}} \\ &\leqslant e^{(\ln C_5)\sum_{i=0}^{\infty}(\frac{m-2}{m})^i + (\ln C_6)\sum_{i=0}^{\infty} i(\frac{m-2}{m})^i} \|\tilde{u}\|_{2\gamma, \tilde{B}_1(Q_0)} \\ &=: C_7 \|\tilde{u}\|_{2\gamma, \tilde{B}_1(Q_0)},\end{aligned}$$

with

$$0 < C_7 \sim \begin{cases} \delta^{-\frac{m}{2\gamma q}}, & b + B > 0, \\ \delta^{-\frac{m}{2\gamma q}(1 - \frac{(\sigma-1)q}{2})}, & b + B = 0, \end{cases} \quad \delta \ll 1.$$

This latter, combined with (19), implies

$$\tilde{u}_0 = \tilde{u}(Q_0) \leqslant \|\tilde{u}\|_{\infty, \tilde{B}_{\frac{1}{2}}(Q_0)} \leqslant C_7 \left(\int_{\tilde{B}_1(Q_0)} \tilde{u}^{2\gamma}\, d\tilde{v}_M\right)^{\frac{1}{2\gamma}} \leqslant C_8 \zeta^{\frac{1}{2\gamma}},$$

with

$$C_8 = C_7 C_2^{\frac{1}{2\gamma}} \sim \begin{cases} \delta^{-1}, & b + B > 0, \\ \delta^{-1 + \frac{m(\sigma-1)}{4\gamma}}, & b + B = 0 \end{cases} \quad \delta \ll 1.$$

Finally, from (18),

$$\left(\varepsilon_0^{-1}\zeta\right)^{\frac{1}{2\gamma}} = 2\tilde{u}_0 \leqslant 2C_8 \zeta^{\frac{1}{2\gamma}},$$

which yields

$$\begin{aligned}\varepsilon_0 &\geqslant (2C_8)^{-2\gamma} = C_9(m, A, B, a, b, \delta, \gamma) \\ &\sim \begin{cases} \delta^{2\gamma}, & b + B > 0, \\ \delta^{2\gamma - \frac{m(\sigma-1)}{2}}, & b + B = 0, \end{cases} \quad \delta \ll 1. \qquad \square\end{aligned} \tag{24}$$

Remark 3. In case the differential inequality (4) has a geometric content then, besides the natural rescaling $\tilde{u} = \lambda^{-q} u$ of u, we have also a rescaling of the coefficients a and b, say $\tilde{a} = \lambda^{-\alpha} a$ and $\tilde{b} = \lambda^{-\beta} b$. Accordingly, q, α, β and σ must be mutually related by $\beta = 2$, $\alpha = 2 + q(1 - \sigma)$. In particular, in these geometric situations, (21) above follows directly from (4).

Proof of Theorem 1. To begin with we consider the case where (6) is satisfied with $\limsup < +\infty$. Assume by contradiction that, for every $D > 2^q$, there exists a sequence of rays $\{R_k^{(D)}\}_{k=1}^{\infty} \subset [1, +\infty)$ such that $R_k^{(D)} \to +\infty$ for $k \to \infty$ and

$$\sup_{M \setminus B_{R_k^{(D)}}} u > \delta_k,$$

where

$$\delta_k := \frac{D}{[R_k^{(D)}]^p}.$$

Then, for every k, we can find $\bar{R}_k^{(D)} > 0$ such that

$$\sup_{A(R_k^{(D)}, R)} u > \frac{D}{[R_k^{(D)}]^p}, \quad \forall R > \bar{R}_k^{(D)}.$$

Choose an integer $n_k \geqslant 3$ such that $n_k R_k^{(D)} > \bar{R}_k^{(D)}$ and set $R_k^{(D)\prime} := \frac{1}{2} R_k^{(D)}$ and $R_k^{(D)\prime\prime} := (2n_k + 1) R_k^{(D)\prime}$.

Let us consider the sequence $\{\epsilon_0^{(k)}\}_{k=1}^{\infty} \subset \mathbb{R}_+$ defined by

$$\epsilon_0^{(k)} := \int_{M \setminus B_{R_k^{(D)\prime}}} u^{2\gamma}\, dv_M$$

so that

$$\int_{A(R_k^{(D)\prime}, R_k^{(D)\prime\prime})} u^{2\gamma}\, dv_M \leqslant \epsilon_0^{(k)}.$$

We have

$$\begin{aligned}
\big[R_k^{(D)\prime}\big]^q \sup_{A(2R_k^{(D)\prime}, R_k^{(D)\prime\prime} - R_k^{(D)\prime})} u &\geqslant \big[R_k^{(D)\prime}\big]^q D \frac{1}{[R_k^{(D)}]^p} \\
&> \big[R_k^{(D)}\big]^{q-p} 2^{-q} D \\
&> 1 \\
&\geqslant \big(\epsilon_0^{(k)}\big)^{-\frac{1}{2\gamma}} \bigg(\int_{A(R_k^{(D)\prime}, R_k^{(D)\prime\prime})} u^{2\gamma}\, dv_M \bigg)^{\frac{1}{2\gamma}}.
\end{aligned}$$

Applying Theorem 1 with $R_1 = r = R_k^{(D)\prime}$, $R_2 = R_k^{(D)\prime\prime}$, $\epsilon_0 = \epsilon_0^{(k)}$ and $\delta = \delta_k$ and recalling estimate (24), we obtain

$$\begin{aligned}
\big[R_k^{(D)}\big]^{p(2\gamma-\xi)} \int_{M \setminus B_{R_k^{(D)\prime}}} u^{2\gamma} &= \big[R_k^{(D)}\big]^{p(2\gamma-\xi)} \epsilon_0^{(k)} \\
&\geqslant \big[R_k^{(D)}\big]^{p(2\gamma-\xi)} C_{10} \delta_k^{(2\gamma-\xi)} \\
&= C_{10} D^{(2\gamma-\xi)},
\end{aligned}$$

for k large enough and for some positive constant C_{10} which does not depend on δ. Since D is arbitrarily large, this contradicts assumption (6), completing the first part of the proof.

If assumption (6) holds in the case $\limsup = 0$, we can reproduce the above proof with $D = 2^{q+1}$, to obtain

$$\limsup_{R \to \infty} \Big\{ R^p \sup_{M \setminus B_R} u \Big\} \leqslant 2^{q+1}. \tag{25}$$

Now we prove that

$$\limsup_{R \to \infty} \Big\{ R^p \sup_{M \setminus B_R} u \Big\} \leqslant \varepsilon, \quad 0 < \forall \varepsilon \ll 1.$$

To this end, we consider the non-negative function $\hat{u} \in \mathrm{lip}_{\mathrm{loc}}(M)$ defined by $\hat{u} := \alpha u$, where $\alpha = \frac{2^{q+2}}{\varepsilon} > 1$. We note that $\hat{u}$ is a solution of (4) satisfying the same assumptions of u. Indeed, obviously, $\hat{u} \in L^{2\gamma}$ and, if $\widetilde{\langle , \rangle} = \lambda^2 \langle , \rangle$, $\lambda = \mathrm{const.} > 0$, then

$$\tilde{\hat{u}} = \alpha \tilde{u} = \alpha \frac{u}{\lambda^q} = \frac{\hat{u}}{\lambda^q}.$$

Moreover

$$\Delta \hat{u} = \alpha \Delta u \geqslant -a\alpha u^\sigma - b\alpha u = -a \frac{\hat{u}^\sigma}{\alpha^{\sigma-1}} - b\hat{u} \geqslant -a\hat{u}^\sigma - b\hat{u}.$$

Finally

$$\limsup_{R \to +\infty} \bigg\{ R^{p(2\gamma-\xi)} \int_{M \setminus B_R} \hat{u}^{2\gamma} \bigg\} = \alpha^{2\gamma} \limsup_{R \to +\infty} \bigg\{ R^{p(2\gamma-\xi)} \int_{M \setminus B_R} u^{2\gamma} \bigg\} = 0.$$

We can then apply (25) to $\hat{u}$, thus obtaining

$$2^{q+1} \geqslant \limsup_{R \to \infty} \Big\{ R^p \sup_{M \setminus B_R} \hat{u} \Big\} = \frac{2^{q+2}}{\varepsilon} \limsup_{R \to \infty} \Big\{ R^p \sup_{M \setminus B_R} u \Big\},$$

that is

$$\limsup_{R\to\infty}\left\{R^p \sup_{M\setminus B_R} u\right\} \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

Since ε is arbitrarily small, this concludes the proof. □

3. Uniform decay estimates under volume conditions

In this section, we consider a solution $u \geqslant 0$ of (4) in the special case $b = 0$ and we show that a slow volume growth of the geodesic balls of M implies a control on the decay at infinity of suitable integral norms of u; see Theorem 4. This fact, in particular, when combined with Theorem 1, will enable us to deduce the rate of uniform decay at infinity of u under volume conditions; see Theorem 6 below. The case of faster volume growths will be also considered. Some remarks in this direction will be given at the end of the section; see Theorem 8 below.

To begin with, elaborating on ideas of M.C. Leung, see Theorem 2.2 in [12], we observe the following

Theorem 4. *Let $(M, \langle,\rangle)$ be a complete Riemannian manifold of dimension $\dim M = m$, and let $u > 0$ be a solution of the equation*

$$\Delta u + a(x)u^{\sigma} = 0, \quad \text{on } M \setminus K, \tag{26}$$

for some compact set K, where $\sigma > 1$ and $a(x)$ is a continuous function satisfying

$$a_1 \leqslant a(x), \quad \text{on } M \setminus K,$$

for some constant $a_1 > 0$. Assume that,

$$\operatorname{vol} B_R(o) \leqslant C_1 R^{\nu}, \quad R \gg 1,$$

for some constants $C_1 > 0$ and

$$\nu < \frac{2\sigma}{\sigma - 1}. \tag{27}$$

Then, for every

$$0 < \alpha < \min\{1; \sigma - \nu(\sigma - 1)/2\}, \tag{28}$$

$u \in L^{\sigma-\alpha}(M)$ and

$$\int\limits_{M\setminus B_R(o)} u^{\sigma-\alpha} \leqslant C_2 R^{\nu - 2\frac{\sigma-\alpha}{\sigma-1}}, \quad R \gg 1, \tag{29}$$

holds for a suitable constant $C_2 > 0$.

Remark 5. As it will be clear from the proof, Eq. (26) can be replaced by the inequality $\Delta u + a(x)u^{\sigma} \leqslant 0$.

Proof of Theorem 4. Let $0 \leqslant \rho \in C_c^{\infty}(M)$ be a cut-off function supported outside the set K. Having fixed $0 < \alpha < 1$ whose value will be specified later, let

$$d > \frac{2(\sigma - \alpha)}{\sigma - 1}.$$

Multiplying both sides of (26) by ρ^d/u^{α}, integrating by parts and elaborating exactly as in the proof of Theorem 2.2 in [12], we obtain that, for any sufficiently small $0 < \varepsilon \ll 1$,

$$0 < \left(a_1 - \frac{\varepsilon d^2(1-\alpha)}{4\alpha(\sigma-\alpha)}\right)\int \rho^d u^{\sigma-\alpha} \leqslant \left(\frac{d^2(\sigma-1)\varepsilon^{-\frac{1-\alpha}{\sigma-1}}}{4\alpha(\sigma-\alpha)}\right)\int |\nabla\rho|^{2\frac{\sigma-\alpha}{\sigma-1}} \rho^{(d\frac{\sigma-1}{\sigma-\alpha}-2)\frac{\sigma-\alpha}{\sigma-1}}. \tag{30}$$

For every $R \gg 1$, we now choose $0 \leqslant \rho = \rho_R \leqslant 1$ in such a way that

$$\text{(a)}\quad \rho = 1 \quad \text{on } B_{3R}\setminus B_{2R}, \qquad \text{(b)}\quad \rho = 0 \quad \text{off } B_{4R}\setminus B_R, \qquad \text{(c)}\quad |\nabla\rho| \leqslant \frac{2}{R} \quad \text{on } M.$$

Inserting into (30) gives

$$\int\limits_{B_{3R}\setminus B_{2R}} u^{\sigma-\alpha} \leqslant C R^{\nu - 2\frac{\sigma-\alpha}{\sigma-1}}. \tag{31}$$

In particular, for every $i \in \mathbb{N}$, applying (31) with $R_i = (3/2)^i R$, we have

$$\begin{aligned}
\int_{B_{3R_j}\setminus B_{2R}} u^{\sigma-\alpha} &= \sum_{i=0}^{j} \int_{B_{3R_i}\setminus B_{2R_i}} u^{\sigma-\alpha} \\
&\leqslant C' R^{\nu-2\frac{\sigma-\alpha}{\sigma-1}} \sum_{i=0}^{j} \left(\frac{3}{2}\right)^{(\nu-2\frac{\sigma-\alpha}{\sigma-1})i} \\
&\leqslant C'' R^{\nu-2\frac{\sigma-\alpha}{\sigma-1}},
\end{aligned}$$

provided α is chosen in such a way that

$$\nu - 2\frac{\sigma-\alpha}{\sigma-1} < 0.$$

Such a condition is satisfied in view of assumptions (27) and (28). To conclude, we now let $j \to +\infty$. $\square$

Combining Theorem 4 with Theorem 1 we deduce the following result. Geometric applications to the non-compact Yamabe problem will be given in Section 4.2 below.

Theorem 6. *Let $(M, \langle , \rangle)$ be a complete Riemannian manifold of dimension $\dim M = m$, enjoying the Sobolev type inequality (1), outside a compact set $K \subset M$. Let $u > 0$ be a function which rescales as in (3), for some $q \geqslant 1$. Assume that u is a solution of the equation*

$$\Delta u + a(x)u^{\sigma} = 0, \quad \text{on } M \setminus K,$$

where $a(x)$ is a continuous function satisfying

$$a_1 \leqslant a(x) \leqslant a_2, \quad \text{on } M \setminus K, \tag{32}$$

for some constants $a_1, a_2 > 0$, and

$$1 < \sigma \leqslant \frac{q+2}{q}.$$

Assume that

$$\operatorname{vol} B_R(o) \leqslant C R^{\nu}, \quad R \gg 1, \tag{33}$$

for suitable constants $C > 0$ and

$$\nu < \frac{2\sigma}{\sigma-1}.$$

If $u \in L^{2\gamma}(M)$, for some

$$\gamma \geqslant \frac{m}{2q},$$

then

$$\lim_{R\to+\infty} R^p \sup_{M\setminus B_R(o)} u = 0, \tag{34}$$

for every $p \leqslant q$ such that either

$$0 \leqslant p\left\{\max\left\{\frac{m}{q}; \sigma\right\}\right\} < \frac{2\sigma - \nu(\sigma-1)}{\sigma-1}, \tag{35}$$

if $B > 0$ or

$$0 \leqslant p\left\{\max\left\{\frac{m}{q}; \sigma\right\} - \frac{(\sigma-1)m}{2}\right\} < \frac{2\sigma - \nu(\sigma-1)}{\sigma-1}, \tag{36}$$

if $B = 0$.

Remark 7. As we noted in the Introduction, in case

$$B=0, \qquad \gamma=\frac{m}{2q}, \qquad 1=q(\sigma-1)/2,$$

conclusion (34) holds with $p=q$, regardless of the validity of the volume assumption (33).

Proof of Theorem 6. Since all the assumptions of Theorem 4 are satisfied, we deduce that $u\in L^{\sigma-\alpha}(M)$ and (29) holds. On the other hand, by (32) we have that u is a solution of (4) with $a=a_2$, $b=0$. Since $u\in L^{2\gamma}(M)$, applying Theorem 1, we conclude that

$$\lim_{R\to+\infty}\sup_{M\setminus B_R} u=0.$$

Set

$$\tilde{\gamma}=\max\left\{\frac{m}{2q};\frac{\sigma-\alpha}{2}\right\}.$$

Then, we can find $\hat{R}>0$ such that $u\leqslant 1$ on $M\setminus B_{\hat{R}}$ and, in view of (29),

$$\int_{M\setminus B_R} u^{2\tilde{\gamma}}\leqslant\int_{M\setminus B_R} u^{\sigma-\alpha}\leqslant GR^{\nu-2\frac{\sigma-\alpha}{\sigma-1}},$$

for every $R\geqslant\hat{R}$ and for some constant $G>0$. As in Theorem 1, let $\xi=\frac{(\sigma-1)m}{2}$ if $B=0$ and $\xi=0$ otherwise. It follows that

$$\limsup_{R\to+\infty} R^{p(2\tilde{\gamma}-\xi)}\int_{M\setminus B_R} u^{2\tilde{\gamma}}\leqslant G\lim_{R\to+\infty} R^{p(2\tilde{\gamma}-\xi)+\nu-2\frac{\sigma-\alpha}{\sigma-1}}=0,$$

provided

$$p(2\tilde{\gamma}-\xi)<2\frac{\sigma-\alpha}{\sigma-1}-\nu \tag{37}$$

and α is chosen as in (28). Note that, for every $0\leqslant p\leqslant q$, (35) implies (37) for α small enough. Therefore, Theorem 1 applies and gives

$$\lim_{R\to+\infty} R^p\sup_{M\setminus B_R} u=0. \qquad \square$$

In Theorem 4 we obtained integral estimates for solutions of (4) with $b=0$, provided the volume growth of the geodesic balls of M was sufficiently slow. In a second step, Theorem 6, we assumed the validity of a Sobolev inequality and got uniform decay conclusions. As a matter of fact, the validity of a Sobolev inequality (without potential terms) can be used directly to get similar results. This follows from the next theorem which is a variation of Proposition 3.1 in [16]; see also Theorem 9.12 in [15] for the case of Sobolev inequalities with non-constant potentials.

Theorem 8. *Let $(M,\langle,\rangle)$ be a complete, non-compact Riemannian manifold satisfying the Sobolev inequality, outside a compact set $K\subset M$,*

$$\left(\int\varphi^{2\mu}\right)^{\frac{1}{\mu}}\leqslant A^2\int|\nabla\varphi|^2, \tag{38}$$

for some constants $A>0$, $\mu>1$ and for every $\varphi\in C_c^\infty(M\setminus K)$. Let $u\geqslant 0$ be a non-null weak solution of the differential inequality

$$\Delta u+au^\sigma+T\frac{|\nabla u|^2}{u}\geqslant 0, \quad \text{on } M\setminus K,$$

for some $\sigma>1$ and $a>0$, $T\in\mathbb{R}$ such that

$$\frac{\mu(\sigma-1)}{\mu-1}-T-1>0.$$

Assume that $u\in L^{2\gamma}(M)$, with $2\gamma=\frac{(\sigma-1)\mu}{\mu-1}$. Then, $u\in L^{2\mu\gamma}(M)$ and

$$\int_{M\setminus B_R} u^{2\mu\gamma}\leqslant\frac{C}{R^{2\mu}}, \tag{39}$$

for some constant $C>0$ and for every $R\gg 1$.

Remark 9. It is known from work by G. Carron [4], that the Sobolev inequality (38) holds, with a different constant, on all of M. Furthermore, according to [5], Sobolev inequality (38) implies the volume growth condition $\operatorname{vol} B_R(o) \geqslant C R^{\frac{2\mu}{\mu-1}}$. Therefore, in a sense, Theorem 8 is complementary to Theorem 4.

Proof of Theorem 8. Let $R_0 > 0$ be so large that $K \subset B_{R_0}$. Since $u \in L^{2\gamma}(M)$, up to choosing R_0 large enough, we can assume that

$$\Bigl(\int_{M\setminus B_{R_0}} u^{\frac{(\sigma-1)\mu}{\mu-1}}\Bigr)^{\frac{\mu-1}{\mu}} < A^{-2}\frac{4(\frac{\mu(\sigma-1)}{\mu-1} - T - 1)}{a(\frac{\mu(\sigma-1)}{\mu-1})^2}.$$

Arguing as in Proposition 3.1 of [16] with the choices (the left-hand symbols refer to [16])

$$\alpha = \frac{\mu-1}{\mu}, \qquad S(\alpha) = A^2, \qquad q(x) = au(x)^{\sigma-1}, \qquad \psi = u, \qquad \sigma = \frac{(\sigma-1)\mu}{\mu-1}, \qquad A = -T,$$

we deduce, for every $\phi \in C_c^\infty(M\setminus B_{R_0})$,

$$\begin{aligned} 0 &< \Bigl\{A^{-2} - C_{\varepsilon,\delta} a\Bigl(\int_{M\setminus B_{R_0}} u^{\frac{(\sigma-1)\mu}{\mu-1}}\Bigr)^{\frac{\mu-1}{\mu}}\Bigr\}\Bigl\{\int (u^\gamma\phi)^{2\mu}\Bigr\}^{\frac{1}{\mu}} \\ &\leqslant \bigl\{\varepsilon^{-1}C_{\varepsilon,\delta} + 1 + \delta^{-1}\bigr\}\int u^{2\gamma}|\nabla\phi|^2, \end{aligned}$$

where, for any $0 < \varepsilon, \delta \ll 1$, we have set

$$C_{\varepsilon,\delta} = \frac{(\frac{\mu(\sigma-1)}{\mu-1})^2}{4}\frac{1+\delta}{(\frac{\mu(\sigma-1)}{\mu-1} - T - 1) - \varepsilon}.$$

Now, we choose $\phi = \phi_R$ satisfying

$$\text{(a)}\quad \phi = 1 \quad \text{on } B_{3R}\setminus B_{2R}, \qquad \text{(b)}\quad \phi = 0 \quad \text{off } B_{4R}\setminus B_R, \qquad \text{(c)}\quad |\nabla\phi| \leqslant \frac{2}{R} \quad \text{on } M.$$

Then, the above gives

$$\Bigl\{\int_{B_{3R}\setminus B_{2R}} u^{2\gamma\mu}\Bigr\}^{\frac{1}{\mu}} \leqslant \frac{D_{\varepsilon,\delta}}{R^2}\int_{M\setminus B_{R_0}} u^{2\gamma},$$

for every $R > R_0$ and for some constant $D_{\varepsilon,\delta} > 0$. Iterating this latter as in the proof of Theorem 4, we conclude

$$\Bigl(\int_{M\setminus B_{2R}} u^{2\gamma\mu}\Bigr)^{\frac{1}{\mu}} \leqslant \frac{D'_{\varepsilon,\delta}}{R^2}\int_{M\setminus B_{R_0}} u^{2\gamma},$$

for a suitable $D'_{\varepsilon,\delta} > 0$. This proves the validity of (39). □

4. Geometric applications

In this section, we present different geometric contests where our main estimating theorem applies. We touch the submanifold theory (Simons-type inequalities), the non-compact Yamabe problem (Yamabe equation), and the conformally flat setting (Weitzenbock-type inequalities). Using such a unifying point of view, known results in the literature are extended.

4.1. Submanifolds with parallel mean curvature and finite total curvature in space-forms

For the ease of notation, we shall limit ourselves to consider constant mean curvature hypersurfaces of dimension $m \geqslant 3$. However, with minor changes in the proofs, all the results we shall present hold for parallel mean curvature submanifolds of arbitrary dimension and co-dimension.

Let $\mathbb{M}_c^{m+1}$ be the $(m+1)$-dimensional space-form of constant sectional curvature c. In case $c < 0$, we identify $\mathbb{M}_c^{m+1}$ with its upper half-space model. By an *H-hypersurface* we mean an oriented, complete, m-dimensional manifold $(M, \langle,\rangle)$ isometrically immersed into $\mathbb{M}_c^{m+1}$ with constant mean curvature (function) H. Up to changing the orientation, if necessary, we can always suppose that $H \geqslant 0$. Let $f : M \to \mathbb{M}_c^{m+1}$ denote the aforementioned constant mean curvature immersion. We say that f has a *finite L^p total curvature* if

$$\left|\mathrm{II} - H\langle , \rangle\right| \in L^p(M),$$

where II stands for the second fundamental form of f and $|\ |$ is the norm of the traceless tensor

$$\phi = \mathrm{II} - H\langle , \rangle.$$

We need to recall the following result which easily follows from the L^1 Sobolev inequality by D. Hoffman and J. Spruck [10].

Theorem 10. *Let $f : M^m \to \mathbb{M}_c^n$ be an isometric immersion, with $c \leqslant 0$ and $m \geqslant 3$. Let $\mathbf{H}$ be the mean curvature vector field of f and suppose that $|\mathbf{H}| \leqslant H < +\infty$. Then M enjoys the L^2 Sobolev inequality*

$$\|\varphi\|^2_{L^{\frac{2m}{m-2}}(M)} \leqslant S_2^2 \|\nabla\varphi\|^2_{L^2(M)} + S_2^2 H^2 \|\varphi\|^2_{L^2(M)}, \tag{40}$$

for some constant $S_2 = S_2(m)$ and for every $\varphi \in C_c^\infty(M)$.

Remark 11. It is a direct consequence of Theorem 10 that each Riemannian product $M \times \mathbb{R}^k$, M compact, supports the L^2 Sobolev inequality (40) for a suitable $H \geqslant 0$. Indeed, according to Nash, fix an isometric immersion $g : M \to \mathbb{R}^N$. Since M is compact, the isometric immersion $g \times id : M \times \mathbb{R}^k \to \mathbb{R}^{N+k}$ has a bounded mean curvature, so that Theorem 10 applies.

Theorem 1 specifies to the case of H-hypersurfaces as follows.

Theorem 12. *Let $f : M^m \to \overline{M}_c^{m+1}$ be a complete, oriented, m-dimensional H-hypersurface in the space-form $\overline{M}_c^{m+1}$ of non-positive sectional curvature $c \leqslant 0$. Let $\phi = \mathrm{II} - H\langle , \rangle$ be the traceless second fundamental form of f. Then*

$$\limsup_{R\to\infty}\Big\{R^p \sup_{M\setminus B_R} |\phi|\Big\} < +\infty \quad (\textit{resp.} = 0),$$

provided

$$\limsup_{R\to+\infty}\Big\{R^{p\tau} \int\limits_{M\setminus B_R} |\phi|^{2\gamma}\Big\} < +\infty \quad (\textit{resp.} = 0), \tag{41}$$

for some $0 \leqslant p \leqslant 1$ and $\gamma \geqslant \frac{m}{2}$, where

$$\tau = \begin{cases} 2\gamma & \text{if } c < 0, \\ 2\gamma - m & \text{if } c = 0 \text{ and } H = 0. \end{cases}$$

Remark 13. Clearly, the above estimates hold merely assuming that the hypersurface at hand has a constant mean curvature outside a sufficiently large compact set.

Remark 14. The case $c = 0$ and $H \neq 0$ has not been considered because, according to [3,18,19], H-hypersurfaces in the Euclidean space with finite $L^{2\gamma}$ total curvature are necessarily minimal; see also the discussion following the proof of the theorem.

Remark 15. In the special case $c = 0$ and $\gamma = m/2$, condition (41) is automatically met and the conclusion of Theorem 12 reads

$$\sup_{M\setminus B_R} |\mathrm{II}| = o\left(\frac{1}{R}\right), \quad \text{as } R \to +\infty.$$

We have thus recovered a well-known estimate by M. Anderson [2].

Proof of Theorem 12. Let $f : M \to \mathbb{M}_c^{m+1}$ be an H-hypersurface, with $c \leqslant 0$, $H \geqslant 0$ and $m \geqslant 3$. From Theorem 10, M enjoys the L^2 Sobolev inequality (40) for every $\varphi \in C_c^\infty(M)$. It is also known that the traceless second fundamental form of f satisfies a Simons type inequality from which one deduces that the locally Lipschitz function $u = |\phi| \geqslant 0$ satisfies

$$\Delta u + (mc)u + \left(1 + \frac{(m-2)^2}{4(m-1)}\right)u^3 \geqslant 0,$$

pointwise where $\phi \neq 0$ and weakly on all of M; see [1,3,18].

For any fixed $\lambda > 0$, let $\tilde{M}$ denote the manifold M endowed with the stretched metric $\widetilde{\langle,\rangle} = \lambda^2 \langle,\rangle$. Then $\tilde{f} = \lambda f : \tilde{M} \to \mathbb{M}_c^{m+1}$ is an $\tilde{H}$-hypersurface with

$$\tilde{H} = \lambda^{-1} H.$$

Note also that

$$\tilde{u} = |\tilde{\phi}|_{\tilde{M}} = \frac{1}{\lambda}|\phi|_M = \frac{1}{\lambda}u.$$

Therefore, to conclude, we can apply Theorem 1 with the choices $u = |\phi|$ and $q = 1$. □

It is a by now well-known general philosophy that, under reasonable requirements, the non-compactness of the H-hypersurface forces the mean curvature H to be small compared with the curvature of the ambient space. More precisely, suppose we are given an isometrically immersed hypersurface $f : M \to \mathbb{M}_c^{m+1}$, $c \leqslant 0$, with constant mean curvature H (outside a compact set). Suppose also that f has a finite L^p total curvature for some $p \geqslant m$. If M is non-compact, then

$$H^2 + c \leqslant 0.$$

In particular, if $c = 0$, then $H = 0$, i.e., the immersion is minimal (at infinity); see [3,19]. Indeed, since the H-hypersurface at hand has a finite L^p total curvature, some $p \geqslant m$, by Theorem 12 above we have

$$\sup_{M \setminus B_R} |\phi| \to 0, \quad \text{as } R \to +\infty.$$

Therefore, as shown in Lemma 2.1 of [7], we can apply a result by P.-F. Leung [13], to conclude the Ricci curvature estimate

$$Ric \geqslant (m-1)\delta^2 \quad \text{on } M \setminus B_{R_0},$$

for some $R_0 > 0$ and some $\delta > 0$. The asserted property is now achieved as a very special case of well-known extensions of the Bonnet–Myers theorem, see [8].

In case the ambient space-form $\mathbb{M}_c^{m+1}$ is hyperbolic, namely, if $c < 0$, we are able to deduce information on the structure at infinity of M provided the mean curvature H is small enough. Recall that an *end* of M with respect to a compact set $\Omega \subset M$ is any of the unbounded connected components of $M \setminus \Omega$. By the geodesic completeness assumption, $M \setminus \Omega$ has only a finite number of such unbounded components, say $n(\Omega)$. Clearly, $n(\Omega)$ monotonically increases with Ω. In case $n(\Omega)$ stabilizes to a finite integer $k < +\infty$ as $\Omega \nearrow M$ exhausts M, we say that M has *finitely many ends*.

Classical minimal submanifolds theory suggests that a finiteness result for the number of ends could be achieved once we have a suitable pointwise control of the second fundamental tensor (curvature estimates), see e.g. [2]. In fact we are able to prove the following

Theorem 16. *Let $f : M^m \to N^n$ be an isometric immersion of the complete, connected m-dimensional Riemannian manifold $(M, \langle,\rangle)$ into the n-dimensional Cartan–Hadamard manifold N of sectional curvature ${}^N\mathrm{Sect} \leqslant c < 0$. Suppose the radial second fundamental form satisfies*

$$\left|\mathrm{II}_x(\nabla r, \nabla r)\right| \leqslant L\bigl(r(x)\bigr),$$

where $r(x)$ is the distance function from some fixed origin $o \in M$ and $L(s) : [0, +\infty) \to [0, \sqrt{-c}]$ is such that

$$\lim_{s \to +\infty} \left\{ \sqrt{-c}s - \int_{R_0}^{s} L(t)\, dt \right\} = +\infty, \tag{42}$$

for some $R_0 > 0$. Then the immersion f is proper, and M has at most a finite number of ends, say $E_1, \ldots, E_k$. Moreover, setting $\mathbf{H}$ for the mean curvature vector field, if

$$\sup |\mathbf{H}| = h < +\infty$$

with

$$(m-1)\sqrt{-c} - mh > 0,$$

then, for each $i = 1, \ldots, k$,

$$Ae^{BR} \leqslant \mathrm{vol}(B_R \cap E_i), \quad R \gg 1, \tag{43}$$

where $A, B > 0$ are suitable constants depending only on m, h and c and B_R denotes the geodesic ball of M centered at o and of radius R.

Remark 17. If $L(s) \equiv L < \sqrt{-c}$, then condition (42) is automatically met. This situation, in the special case N is the hyperbolic space, has been previously considered in the paper [6] by P. Castillon. Note also that, even in this special situation, the volume growth estimate of the ends is new.

We note that, combining Theorem 16 with the curvature estimates of Theorem 12 and using Gauss equations and the Petersen–Wei integral comparison theorem (see [14], and Chapter 2.2 in [15]), we immediately get the next

Theorem 18. *Suppose the H-hypersurface $f: M \to \mathbb{M}_c^{m+1} = \mathbb{H}_c^{m+1}$, $c < 0$, has a finite L^p total curvature, for some $p \geqslant m$. If*

$$H < \frac{m-1}{m}\sqrt{-c},$$

then the immersion f is proper, and M has at most a finite number of ends $E_1, \ldots, E_k$ whose volumes satisfy

$$A_1 e^{B_1 R} \leqslant \operatorname{vol}(B_R \cap E_i) \leqslant A_2 e^{B_2 R}, \quad R \gg 1,$$

where $A_1, A_2, B_1, B_2 > 0$ are suitable constants depending only on H, m and c.

Proof of Theorem 16. We set $\rho(y) = \operatorname{dist}_N(y, f(o))$, where $o \in M$ is the reference point in the statement of the theorem. Moreover we define functions

$$\begin{aligned}
\operatorname{sn}_c(t) &:= \frac{1}{\sqrt{-c}} \sinh(\sqrt{-c}t), \\
\operatorname{cn}_c(t) &= \dot{\operatorname{sn}}_c(t), \\
\operatorname{in}_c(t) &= \int_0^t \operatorname{sn}_c(q)\, dq.
\end{aligned}$$

Let γ be any unit-speed minimizing geodesic issuing from $\gamma(0) = o$. By the composition law of the Hessians, we have

$${}^M\operatorname{Hess}\bigl(\operatorname{in}_c\bigl(\rho(f)\bigr)\bigr) = {}^N\operatorname{Hess}\bigl(\operatorname{in}_c(\rho)\bigr)(df, df) + d\bigl(\operatorname{in}_c(\rho)\bigr)(\mathrm{II}).$$

Furthermore, by Hessian comparison,

$${}^N\operatorname{Hess}\bigl(\operatorname{in}_c(\rho)\bigr) \geqslant \operatorname{cn}_c(\rho)\langle , \rangle_N.$$

Therefore, on noting that

$$\bigl|\nabla\bigl(\operatorname{in}_c(\rho)\bigr)\bigr|_N = \bigl|\operatorname{sn}_c(\rho)\nabla\rho\bigr|_N = \operatorname{sn}_c(\rho),$$

we get

$$\begin{aligned}
\frac{d^2}{ds^2} \operatorname{in}_c\bigl(\rho\bigl(f(\gamma)\bigr)\bigr) &= {}^M\operatorname{Hess}\bigl(\operatorname{in}_c\bigl(\rho(f)\bigr)\bigr)(\dot\gamma, \dot\gamma) \\
&\geqslant \operatorname{cn}_c\bigl(\rho\bigl(f(\gamma)\bigr)\bigr) - \operatorname{sn}_c\bigl(\rho(f)\bigr)\bigl|\mathrm{II}(\dot\gamma, \dot\gamma)\bigr| \\
&\geqslant \operatorname{cn}_c\bigl(\rho\bigl(f(\gamma)\bigr)\bigr) - \operatorname{sn}_c\bigl(\rho\bigl(f(\gamma)\bigr)\bigr)L\bigl(r(\gamma)\bigr).
\end{aligned}$$

Whence, integrating over $[R_0, s]$, we obtain

$$\begin{aligned}
\frac{d}{ds} \operatorname{in}_c\bigl(\rho\bigl(f(\gamma)\bigr)\bigr) &\geqslant \int_{R_0}^{s} \bigl\{\operatorname{cn}_c\bigl(\rho\bigl(f\bigl(\gamma(t)\bigr)\bigr)\bigr) - \operatorname{sn}_c\bigl(\rho\bigl(f\bigl(\gamma(t)\bigr)\bigr)\bigr)L(t)\bigr\}\, dt + d \\
&\geqslant \int_{R_0}^{s} \frac{1}{\sqrt{-c}} \cosh(\sqrt{-c}t)\bigl\{\sqrt{-c} - \tanh(\sqrt{-c}t)L(t)\bigr\}\, dt + d \\
&\geqslant \frac{1}{\sqrt{-c}} \int_{R_0}^{s} \bigl\{\sqrt{-c} - \tanh(\sqrt{-c}t)L(t)\bigr\}\, dt + d,
\end{aligned}$$

with

$$d = d\bigl(c, \gamma(R_0)\bigr) = \frac{d}{ds} \operatorname{in}_c\bigl(\rho \circ f \circ \gamma(R_0)\bigr).$$

Note that

$$\frac{d}{ds}\operatorname{in}_c\bigl(\rho\bigl(f(\gamma)\bigr)\bigr) = \bigl\langle\nabla \operatorname{in}_c\bigl(\rho(f)\bigr), \dot\gamma\bigr\rangle = \operatorname{sn}_c\bigl(\rho\bigl(f(\gamma)\bigr)\bigr)\bigl\langle\nabla\rho(f), \dot\gamma\bigr\rangle,$$

where

$$\bigl\langle\nabla\rho(f), \dot\gamma\bigr\rangle \leqslant \bigl|\nabla\rho(f)\bigr|\circ\gamma \leqslant 1. \tag{44}$$

Therefore

$$\operatorname{sn}_c\bigl(\rho\bigl(f(\gamma)\bigr)\bigr) \geqslant \operatorname{sn}_c\bigl(\rho\bigl(f(\gamma)\bigr)\bigr)\bigl\langle\nabla\rho(f), \dot\gamma\bigr\rangle \geqslant \frac{1}{\sqrt{-c}}\int_{R_0}^{s}\bigl\{\sqrt{-c} - \tanh(\sqrt{-c}t)L(t)\bigr\}\,dt + d. \tag{45}$$

For any $x \in M \setminus B_{R_0}$, choose a minimizing γ issuing from $\gamma(0) = o$. Then, $x = \gamma(s)$ with $s = r(x) > R_0$ and assumption (42) ensures that the integral in (45) diverges as s tends to infinity. Thus we get

$$f(x) \to \infty, \quad \text{as } x \to \infty,$$

i.e. f is a proper map.

As above, for any $x \in M \setminus B_{R_0}$, choose a minimizing γ from $\gamma(0) = o$ to $\gamma(r(x)) = x$ and note that, by (44) and (45), we have

$$\bigl|\nabla\rho(f)\bigr|(x) = \bigl|\nabla\rho(f)\bigr|\circ\gamma\bigl(r(x)\bigr) \geqslant \bigl\langle\nabla\rho(f), \dot\gamma\bigr\rangle \geqslant \frac{\frac{1}{\sqrt{-c}}\int_{R_0}^{r(x)}\{\sqrt{-c} - \tanh(\sqrt{-c}t)L(t)\}\,dt + d}{\operatorname{sn}_c(\rho(f(\gamma)))} > 0,$$

provided $r(x) > R_0$ is large enough. According to previous discussions, $\rho(f)(x)$ is a non-negative, smooth, proper function satisfying

$$\nabla\rho(f) \neq 0 \quad \text{on } M \setminus B_{R_0}.$$

We deduce the finiteness of the ends by an application of the following classical lemma of Morse theory.

Lemma 19. *Let $(M, \langle,\rangle)$ be a complete Riemannian manifold. Suppose that M supports a smooth, exhaustion function $f : M \to \mathbb{R}$ satisfying $|\nabla f| \neq 0$ on $M \setminus B_{\bar R}(o)$ for some $\bar R > 0$. Then, there exists $R_1 > \bar R$ such that $f^{-1}([R_1, +\infty))$ is a half cylinder over $f^{-1}(R_1)$, i.e., it is diffeomorphic to $f^{-1}(R_1) \times [0, +\infty)$.*

We now proceed to prove the volume growth estimates. To this purpose, we suppose that M has only one end. Since the alluded estimates do not depend on the center of the ball, up to changing the constants, the many-ends case follows easily using a packing argument.

As above, by the composition law of tension fields and Hessian comparison, since f is isometric we get

$${}^{M}\Delta(\operatorname{in}_c\circ\rho\circ f) \geqslant m\bigl\{\operatorname{cn}_c\bigl(\rho(f)\bigr) - h\operatorname{sn}_c\bigl(\rho(f)\bigr)\bigr\}.$$

Integrating and applying the divergence theorem give

$$m\int_{B_R(o)}\bigl\{\operatorname{cn}_c\bigl(\rho(f)\bigr) - h\operatorname{sn}_c\bigl(\rho(f)\bigr)\bigr\} \leqslant \int_{\partial B_R(o)}\operatorname{sn}_c\bigl(\rho(f)\bigr).$$

Whence, we obtain

$$m\left(1 - h\frac{\operatorname{sn}_c(R)}{\operatorname{cn}_c(R)}\right)\int_{B_R(o)}\operatorname{cn}_c\bigl(\rho(f)\bigr) \leqslant \frac{\operatorname{sn}_c(R)}{\operatorname{cn}_c(R)}\int_{\partial B_R(o)}\operatorname{cn}_c\bigl(\rho(f)\bigr)$$

that is

$$m\frac{\operatorname{cn}_c(R)}{\operatorname{sn}_c(R)} - mh \leqslant \frac{\int_{\partial B_R(o)}\operatorname{cn}_c(\rho(f))}{\int_{B_R(o)}\operatorname{cn}_c(\rho(f))}.$$

The desired inequality (43) now follows integrating the latter on $[R_0, R]$, using the co-area formula, noting that $1 \leqslant \operatorname{cn}_c(\rho(f(x))) \leqslant \operatorname{cn}_c(R)$ on B_R and, finally, letting $R_0 \to 0$. □

4.2. The non-compact Yamabe problem and conformal immersions

Let $(M, \langle,\rangle)$ be an m-dimensional manifold, $m \geqslant 3$. Denote by $s(x)$ its scalar curvature. The Riemannian metric $\widetilde{\langle,\rangle}$ is obtained by $\langle,\rangle$ through a pointwise conformal deformation if $\widetilde{\langle,\rangle} = v^2(x)\langle,\rangle$ for some smooth function $v(x) > 0$. Given an immersion $f : (M, \langle,\rangle_M) \to (N, \langle,\rangle_N)$, between Riemannian manifolds of dimensions $\dim M = m \leqslant n = \dim N$, we say that f is conformal if the pull back metric $\widetilde{\langle,\rangle}_M = f^*\langle,\rangle_N$ is pointwise conformal to the background metric $\langle,\rangle_M$. Setting as above $\widetilde{\langle,\rangle}_M = v^2\langle,\rangle_M$, we have

$$v^2 = \frac{1}{m}\operatorname{trace}_{\langle,\rangle_M}\left(f^*\langle,\rangle_N\right) = \frac{1}{m}|df|^2_M,$$

the energy density of the map f.

Classically, one defines $u(x) = v^{\frac{m-2}{2}}(x)$ and verifies that the conformal factor $u(x)$ satisfies the semi-linear elliptic differential equation (called the Yamabe equation)

$$c_m\Delta u - s(x)u + k(x)u^{\frac{m+2}{m-2}} = 0, \tag{46}$$

where c_m is a constant which depends only on the dimension m and $k(x)$ is the scalar curvature of the deformed metric $\widetilde{\langle,\rangle}$. A challenging problem is to establish existence and estimating results for positive solutions $u(x)$ of (46) when the deformed scalar curvature $k(x)$ is positive or changes its sign. In this direction, relatively few results are known; for comparison with our main theorem, see e.g. [20] by Q.S. Zhang, and [11] by S.T. Kim.

Note that, in case

$$c_m^{-1}s(x) \geqslant -b, \qquad c_m^{-1}k(x) \leqslant a, \quad \text{on } M \setminus K, \tag{47}$$

for some compact set K and some constants $a, b > 0$, (46) implies that u is a solution of inequality (4), with $\sigma = (m+2)/(m-2)$. On the other hand, it is easy to verify that a rescaling of the original metric $\widetilde{\langle,\rangle} = \lambda^2\langle,\rangle$, $\lambda = \text{const.} > 0$, produces a rescaling of the deformation function u of the form

$$\tilde{u}(x) = \frac{u(x)}{\lambda^{\frac{m-2}{2}}},$$

so that, if a Sobolev inequality of the type

$$\left(\int_M \varphi^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leqslant A^2\int_M |\nabla\varphi|^2 + B^2\int_M \varphi^2, \quad \varphi \in C_c^\infty(M \setminus K), \tag{48}$$

holds, we are precisely in the situation described in Theorem 1 and the uniform estimates at infinity of the conformal factor u can be deduced. If, furthermore, the original metric is scalar flat in a neighborhood of infinity (so that the linear term in the Yamabe equation disappears) and the volume growth of the underlying manifold is suitably controlled, we can also apply Theorem 6 and conclude uniform decay estimates under volume conditions. By way of example, we point out the following

Theorem 20. *Let $(M, \langle,\rangle)$ be a complete, Riemannian manifold of dimension $m \geqslant 3$. Suppose that M is scalar-flat outside a compact set $K \subset M$ and that it supports a Sobolev inequality of the type*

$$\left(\int \varphi^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leqslant A^2\int |\nabla\varphi|^2 + B^2\int \varphi^2, \quad \forall\varphi \in C_c^\infty(M \setminus K),$$

with $A > 0$ and $B \geqslant 0$. Assume that

$$\operatorname{vol} B_R(o) \leqslant CR^\nu, \quad R \gg 1,$$

for some fixed origin $o \in M$, and some constants $C, \nu > 0$ with

$$\nu < \frac{m+2}{2}.$$

Suppose that the background metric can be conformally deformed to a new metric $\widetilde{\langle,\rangle} = u^{\frac{4m}{m-2}}\langle,\rangle$ with scalar curvature $k(x)$ satisfying

$$a_1 \leqslant k(x) \leqslant a_2, \quad \text{on } M \setminus K,$$

for some constants $a_1, a_2 > 0$. If the deformation factor $u(x) > 0$ satisfies $u \in L^{2\gamma}(M)$, for some $\gamma \geqslant m/(m-2)$, then

$$\lim_{R\to+\infty} R^p \sup_{M\setminus B_R(o)} u = 0,$$

for every $p \leqslant (m-2)/2$ *such that*

$$0 \leqslant p\frac{2sm}{m-2} < \frac{m+2}{2} - \nu.$$

Remark 21. As we observed in the Introduction, in case $B = 0$ and $\gamma = m/(m-2)$, the uniform estimate holds with $p = (m-2)/2$, regardless of the validity of the volume growth assumption.

Remark 22. It is not difficult to construct an example of a complete manifold N with controlled volume growth, supporting a Sobolev inequality with potential term and scalar flat outside a compact set. For instance, let S be a compact Riemann surface of genus $g \geqslant 2$ endowed with its Poincarè metric of constant Gaussian curvature -1 and let $\mathbb{S}^2$ denotes the standard 2-sphere of constant Gaussian curvature $+1$. Then, according to Remark 11, the Riemannian product $M = S \times \mathbb{S}^2 \times \mathbb{R}^k$ enjoys the desired Sobolev inequality. Furthermore, by the additivity property of the scalar curvature under Riemannian product, M is scalar flat. Finally, M has a polynomial volume growth of order k. By a perturbation of the metric of M on a compact set we get the desired manifold N.

A situation of special interest, where the above Sobolev inequality holds, is represented by locally conformally flat manifolds with bounded above scalar curvature.

Recall that the m-dimensional Riemannian manifold $(M, \langle,\rangle)$ is said to be locally conformally flat if, for every $x \in M$, there is a coordinate chart $\xi : U \to \mathbb{R}^m$ centered at $x \in U$ which induces on $\xi(U)$ a metric $(\xi^{-1})^*\langle,\rangle_M$ conformal to the standard Euclidean metric. If M has a conformal immersion into the standard sphere $\mathbb{S}^m$ then M is locally conformally flat. Conversely, according to a result by N. Kuiper, if M is simply connected and locally conformally flat then M can be conformally immersed into $\mathbb{S}^m$; see [17]. Now, suppose M has a conformal immersion into the standard sphere $\mathbb{S}^m$, say $\phi : M \to \mathbb{S}^m$. Clearly, using the above notations, $k(x) = m(m-1)$ so that the second condition in (47) is met. Define the Yamabe quotient of M as

$$Q(M) = \inf_{\varphi \in C_c^\infty(M)\setminus\{0\}} \frac{\int \{|\nabla\varphi|^2 + c_m^{-1} s(x)\varphi^2\}}{(\int_M \varphi^{\frac{2m}{m-2}})^{\frac{m-2}{m}}},$$

where $s(x)$ is the scalar curvature of $(M, \langle,\rangle)$. Then, it is known from work by R. Schoen and S.T. Yau [17], that $Q(M) = Q(\mathbb{S}^m) > 0$. In the special case the scalar curvature of M is such that $s(x) \leqslant S$, for some $S \geqslant 0$, (48) is satisfied with $A = A(m) = Q(\mathbb{S}^m)^{-1}$ and $B = \sqrt{c_m^{-1} Q(\mathbb{S}^m)^{-1} S}$.

With this preparation, applying Theorem 1, one readily concludes the validity of the following

Theorem 23. *Let* $(M, \langle,\rangle)$ *be a complete, locally conformally flat Riemannian manifold of dimension* $m \geqslant 3$ *whose scalar curvature satisfies*

$$0 \neq \sup_M |s(x)| < +\infty.$$

Let $\phi : M \to \mathbb{S}^m$ *be a conformal immersion of* M *into the standard sphere* $\mathbb{S}$. *If*

$$\limsup_{R\to+\infty} \left\{ R^{2p\gamma} \int_{M\setminus B_R} |d\phi|^{(m-2)\gamma} \right\} < +\infty \quad (\textit{resp.} = 0), \tag{49}$$

for some $0 \leqslant p \leqslant \frac{m-2}{2}$ *and* $\gamma \geqslant \frac{m}{m-2}$, *then*

$$\limsup_{R\to\infty} \left\{ R^{\frac{2p}{m-2}} \sup_{M\setminus B_R} |d\phi| \right\} < +\infty \quad (\textit{resp.} = 0).$$

It is important to realize that the integrability condition (49) implies some restrictions on the geometry of the conformally deformed metric

$$\widetilde{\langle,\rangle} = \phi^*\langle,\rangle = v^2\langle,\rangle.$$

Indeed, set for brevity $\tilde{M} = (M, \widetilde{\langle,\rangle})$. Then, we have the following

Proposition 24. *Let* $(M, \langle,\rangle)$ *be a complete, locally conformally flat Riemannian manifold of dimension* $m \geqslant 3$ *whose scalar curvature satisfies* $\sup_M s(x) < +\infty$. *Let* $\phi : M \to \mathbb{S}^m$ *be a conformal immersion such that* $|d\phi| \in L^m$. *If* M *is not conformally diffeomorphic to the standard sphere then either* $\tilde{M}$ *is not geodesically complete or its scalar curvature is not bounded from above.*

Proof. Since $0 < Q(\mathbb{S}^m) = Q(M) = Q(\tilde{M})$, if the scalar curvature $k(x)$ of $\tilde{M}$ satisfies $\sup_M k(x) < +\infty$ we deduce that also $\tilde{M}$ supports a Sobolev inequality. On the other hand, the integrability assumption of $|d\phi|$ means precisely that $\mathrm{vol}(\tilde{M}) < +\infty$. But it is known that a complete manifold, with finite volume satisfying a Sobolev inequality (clearly with a potential term) must be compact; see Proposition 3.5 in [9]. Standard topological arguments now show that the immersion ϕ is in fact a diffeomorphism. □

4.3. Ricci curvature estimates of conformally flat manifolds with asymptotically constant scalar curvature

Let $(M, \langle,\rangle)$ be a locally conformally flat, m-dimensional manifold which is realized as an immersed submanifold of $\mathbb{S}^m$. Suppose that M has a constant scalar curvature

$$Scal(x) \equiv S, \quad \text{on } M \setminus K,$$

for some compact set $K \subset M$. According to what we have seen in the previous section, M enjoys the Sobolev inequality

$$\left(\int_M \varphi^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leqslant Q(\mathbb{S}^m)^{-1} \int_M |\nabla\varphi|^2 + S\,Q(\mathbb{S}^m)^{-1} \int_M \varphi^2, \quad \forall\varphi \in C_c^\infty(M \setminus K).$$

Obviously, in case $S \leqslant 0$, the Sobolev inequality does not present any potential term. Let

$$T = Ric - \frac{Scal}{m}\langle,\rangle$$

denote the traceless Ricci tensor of M. It is known by standard computations (see e.g. Section 9 in [15]) that the locally Lipschitz function $u = |T| \geqslant 0$ is a weak solution of the Bochner-type differential inequality

$$\Delta u - \frac{S}{m-1}u + \sqrt{\frac{m}{m-1}}u^2 \geqslant \frac{2}{m}\frac{|\nabla u|^2}{u}, \quad \text{on } M \setminus K.$$

In particular, in case $S \geqslant 0$, this latter implies

$$\Delta u + \sqrt{\frac{m}{m-1}}u^2 \geqslant 0, \quad \text{on } M \setminus K,$$

while, in case $S < 0$, we obtain that u solves

$$\Delta u - \frac{S}{m-1}u + \sqrt{\frac{m}{m-1}}u^2 \geqslant 0, \quad \text{on } M \setminus K.$$

Since u rescales by a factor λ^{-2} as the background metric is dilated by a constant factor $\lambda^2 > 0$, we can apply Theorem 1 to deduce the validity of the following estimating result.

Theorem 25. *Let $(M, \langle,\rangle)$ be an $m\,(\geqslant 3)$-dimensional, locally conformally flat manifold which is realized as an immersed submanifold of $\mathbb{S}^m$. Suppose that M has a constant scalar curvature $Scal(x) \equiv S$, outside a compact set $K \subset M$. Then*

$$\limsup_{R\to+\infty} R^p \sup_{M\setminus B_R(o)} \left| Ric - \frac{S}{m}\langle,\rangle \right| < +\infty \quad (\text{resp. } = 0),$$

provided, for some $p \leqslant 2$ and $2\gamma \geqslant m/2$, it holds

$$\limsup_{R\to+\infty} R^{p\tau} \int_{M\setminus B_R(o)} \left| Ric - \frac{S}{m}\langle,\rangle \right|^{2\gamma} < +\infty \quad (\text{resp. } = 0),$$

where, according to the value of S,

$$\tau = \begin{cases} 2\gamma & \text{if } S \neq 0, \\ 2\gamma - \frac{m}{2} & \text{if } S = 0. \end{cases}$$

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